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Arbiter as the Third Man in Classical and Quantum Games

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Abstract We study the possible influence of a not necessarily sincere arbiter on the course of classical and quantum 2×2 games and we show that this influence in the quantum case is much bigger than in the classical case. Extreme sensitivity of quantum games on initial states of quantum objects used as carriers of information in a game shows that a quantum game, contrary to a classical game, is not defined by a payoff matrix alone but also by an initial state of objects used to play a game. Therefore, two quantum games that have the same payoff matrices but begin with different initial states should be considered as different games.

Keywords Quantum games · Symmetric games

1 Introduction

Exchange of information in the course of a game is often performed in a non-verbal way. For example players show cards, flip coins, move figures on a board, etc., which means that they utilize for this purpose really existing physical objects. Observation that even some very simple games can change drastically when players use quantum objects instead of classical ones was the beginning of the theory of quantum games [1–3]. Theory of quantum games is still 'in statu nascendi' and till now the majority of authors studied only the simplest case of static two-person two-strategy games (see, e.g., [4, 5] for a review), i.e., games in which each of two players has to choose one of two possible strategies, and they do it simultaneously not knowing which strategy is chosen by the other player. Such classical

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games are completely defined mathematically by specifying a 2×2 bi-matrix of payoffs that players get when a game is finished, and can be played even by dumb and illiterate players in the following way: An arbiter prepares two coins in the 'heads-up' state and passes one of them to each of the players. Each of the players either does nothing, which means that he rests with his first strategy, or he flips his coin, which means that he chooses his second strategy. Then the coins are passed back to the arbiter who checks the state of both coins (i.e., he makes a measurement!), and announces the result. This way of playing 2×2 games can be easily 'quantized': it is enough to replace coins by two-state quantum objects (qubits), while leaving all the rest of the scheme unchanged. Actually, this way of 'quantizing' the classical Battle of The Sexes game was proposed by Marinatto and Weber in [3] and it is, according to our opinion, the most natural way of 'quantizing' classical games (the scheme proposed by Eisert, Wilkens, and Lewenstein in [2] allows in the quantum case strategies that are not allowed in the classical case, therefore the original game is in fact replaced by another one [6]).

However, when a 2×2 quantum game is played according to the Marinatto-Weber scheme the role of the arbiter is not passive anymore since it is up to him what initial state of two coins/qubits he passes to the players. Marinatto and Weber in [3], as well as other authors writing papers on quantum games, did their best in order to 'save' the original classical game in their quantum scheme: usually the original classical game was recovered from the quantum one when the initial state of two qubits prepared by the arbiter was not entangled.

In this paper we go beyond this constraint and allow the arbiter to be the Third Man in a game. This means that he can cheat the players by sending them coins/qubits in the other initial state than they expect. Since the players do not know the actual state of coins/qubits they get, they choose their strategies according to their *belief*, not according to the *actual state* of coins/qubits that are sent to them. In order to be illustrative, we confine our considerations to symmetric 2×2 games in which no payoffs are identical since any such game falls into one of the three categories [7]: I (Prisoner's Dilemma Game), II (Coordination Game), or III (Hawk-Dove Game). We study what changes of such a game can be induced by a 'classical cheating arbiter', i.e., by the arbiter that can pass to the players two coins in the initial state (H,T), or (T,H), or (T,T), instead of the presumed initial state (H,H). We show that he can only change a symmetric game into a non-symmetric one but he cannot change a category of a game if it remains symmetric. Then, we show that for a 'quantum cheating arbiter' that passes to the players two qubits that are not in the presumed initial state $|00\rangle$ but are in another state that is allowed by quantum mechanics the variety of possible changes is much bigger which allows him to exert his influence on the course of a game in many ways that are impossible for his 'classical' colleague.

2 Symmetric 2 × 2 Games

Any symmetric two-player game is fully characterized by the pure-strategy payoff matrix to one of the players

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{1}$$

since the second player's payoff matrix is the transpose of the matrix A: $B = A^T$. A value of an element a_{ij} of the matrix A is the payoff to the player A when he plays an i-th pure strategy and his opponent plays a j-th strategy.

In the case when each player has only two pure strategies and no payoffs are identical there are only three generic categories of symmetric two-player games. They can be distinguished according to signs of differences $\alpha = a_{11} - a_{21}$ and $\beta = a_{22} - a_{12}$, and are as follows (see, e.g., [7]):

Category I (Prisoner's Dilemma Game): $\alpha\beta < 0$.
Category II (Coordination Game): $\alpha, \beta > 0$.
Category III (Hawk-Dove Game): $\alpha, \beta < 0$.

It occurs that within each category various games differ only with respect to specific values of payoffs but possess the same dominance relations, Nash equilibria, etc., i.e., all games that belong to the same category should be played in the same way.

3 Arbiter as the 'Third Man' in Classical Games

Let us now consider symmetric 2×2 games played in a previously mentioned way that allows an arbiter to be the Third Man in a game. Information between the arbiter and players is conveyed by a pair of classical two-state objects, e.g., two coins. In the beginning the arbiter passes to each of the players one coin in the 'heads-up' state that symbolizes the first strategy. Each of the players can either return to the arbiter his coin without changing its state, which means that he remains with the first strategy, or he can flip his coin informing in this way the arbiter that he chooses the second strategy. Of course this way of exchanging information does not make the game different from the same game played in any other way, although it can be played now even by dumb and illiterate players. However, let us assume that the players are blind and that the arbiter is not sincere and can cheat the players passing them in the beginning of the game two coins that are not in a ('heads-up','heads-up') = (H,H) state but are in another possible state: (H,T), (T,H), or (T,T). In such a situation blind players who do not expect that the arbiter can cheat them will choose their strategies according to their *belief*, not according to the *actual state* of coins they get and this will surely lead to a lot of confusion. We think, however, that the considered problem is not purely artificial and that such situations can be encountered also in the real life, e.g., at the stock exchange where players sometimes do not know what game they are actually playing.

Although the sketched situation might give rise to numerous interesting problems, in this paper we shall concentrate on the following one: what changes of categories of a symmetric 2×2 game in which no payoffs are identical can be made by a 'cheating arbiter'. The answer for a classical arbiter is contained in the following

Theorem 1 *A cheating classical arbiter cannot change a category of symmetric* 2×2 *game in which no payoffs are identical. He can only change such game into a non-symmetric game.*

Proof Let

$$(A, B) = \begin{bmatrix} (x, x) & (y, w) \\ (w, y) & (z, z) \end{bmatrix}, \tag{2}$$

where all numbers x, y, w, z are different, be a bi-matrix of a symmetric 2×2 game. If the arbiter cheats players by sending them two coins that are in the (H,T) state instead of the expected (H,H) state, the second player chooses the second strategy thinking that he has chosen the first strategy and vice versa, so now the bi-matrix of the game becomes

$$(A', B') = \begin{bmatrix} (y, w) & (x, x) \\ (z, z) & (w, y) \end{bmatrix} \tag{3}$$

i.e., it is the original bi-matrix of payoffs with permuted columns. However, since

$$(A')^T = \begin{bmatrix} y & z \\ x & w \end{bmatrix} \neq B' = \begin{bmatrix} w & x \\ z & y \end{bmatrix}, \tag{4}$$

and all payoffs are different, a game characterized by the bi-matrix (A', B') is not symmetric. The same arguments apply when the arbiter sends to the players coins in the (T,H) state instead of the expected state (H,H).

Finally, if the arbiter sends to the players coins in the state (T,T) instead of (H,H), then both players play the 'opposite' strategies than they expect, and the bi-matrix of payoffs becomes

$$(A'', B'') = \begin{bmatrix} (z, z) & (w, y) \\ (y, w) & (x, x) \end{bmatrix}. \tag{5}$$

A game characterized by such bi-matrix of payoffs is symmetric, but if for the original game $\alpha = x - w$, $\beta = z - y$, then for the game characterized by the bi-matrix (A'', B'') $\alpha'' = z - y = \beta$ and $\beta'' = x - w = \alpha$. Therefore, a new game remains in the same category as the original one, which finishes the proof. □

4 Quantum Games Played According to the Marinatto-Weber Scheme

In one of the first papers on quantum games Marinatto and Weber [3] 'quantized' the classical Battle of The Sexes game in a way that essentially boils down to replacing classical carriers of information, e.g., coins mentioned in the previous section, by two-state quantum objects (qubits) while leaving all the rest of the procedure unchanged. According to their idea a 2×2 quantum game should be played as follows: An arbiter prepares a pair of two-state quantum objects in a specific state, e.g., two spins in a spin-up state, and sends one of these objects to each of the players. Each player either returns his object to the arbiter unchanged, i.e., applies to it an identity operation, or applies to the object an operation that changes its state into the opposite state, e.g., in the case of spins he flips the spin. Then the arbiter measures the state of both objects he got back from the players and announces the result taking into account the payoff matrix of the game. It is obvious that the only difference between playing classical 2×2 games in a way described in the previous section and playing quantum 2×2 games according to the Marinatto-Weber scheme is such that in the former information is carried by classical and in the latter by quantum objects.

However, while a pair of classical two-state objects can be only in one of four possible pure states since its space of pure states is a Cartesian product of two two-element sets, a pair of quantum objects possesses an infinity of pure states, in particular various superpositions of 'basic' states that represent pure classical strategies, and even 'less classical' entangled states, which are the main source of differences between classical and quantum games. In particular, when a pair of quantum objects used to play a game is in an entangled state, the well-known EPR correlations introduce a kind of 'unconscious communication' between players that does not exist in the classical case and which allowed Marinatto and Weber to find new solutions of the Battle of The Sexes game.

The way of playing quantum games proposed by Marinatto and Weber is not the only one proposed in the literature, but according to our opinion a quantum game played in this way differs the least from its classical prototype. Finally, the players even do not have to know whether carriers of information used in their game are classical or quantum: they might be informed only that they either should do nothing, or press a button causing in this way a 'flip' of a state of an object sent by the arbiter. Although the majority of authors writing on quantum games seems to follow Eisert-Wilkens-Lewenstein [2] and admit that players may use as an allowed strategy any unitary operation, we rather agree with van Enk and Pike [6] that changing the set of allowed strategies means changing the game itself, and therefore we prefer the scheme proposed by Marinatto and Weber.

5 Arbiter as the 'Third Man' in Quantum Games

In the case of quantum games played according to the Marinatto-Weber scheme a cheating arbiter has more possibilities of exerting his influence on a game since he can send to the players a pair of qubits that is not, as the players believe, in the state $|00\rangle$ but is in any state of the form

$$|\psi_{in}\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle \tag{6}$$

where complex coefficients satisfy the normalization requirement $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$. Let us stress that our aim is exactly opposite to the aim of all authors writing up to now papers on quantum games: while they did their best in order to keep the original classical game as a 'special case' of a quantum game, we want to study what *changes* of a game can be made by a 'cheating quantum arbiter'. Since we would like to stay, however, within the category of symmetric games, we begin with the following

Theorem 2 *If in the initial state* (6) *prepared by an arbiter* $|b| = |c|$, *then a symmetric* 2×2 *quantum game in which no payoffs are identical remains symmetric after the intervention of the arbiter. Moreover, if the payoff matrix* (2) *of a game is such that* $x \neq z$ *or* $y \neq w$, *then the condition* $|b| = |c|$ *is also necessary for preserving symmetry of a game.*

Proof Following calculations of Marinatto and Weber [3] one can check that if the bi-matrix of the original game is (2), then the initial state $|\psi_{in}\rangle$ (6) yields the following payoff matrices for the player A

$$\begin{bmatrix} x|a|^2 + y|b|^2 + w|c|^2 + z|d|^2 & x|b|^2 + y|a|^2 + w|d|^2 + z|c|^2 \\ x|c|^2 + y|d|^2 + w|a|^2 + z|b|^2 & x|d|^2 + y|c|^2 + w|b|^2 + z|a|^2 \end{bmatrix} \tag{7}$$

and for the player B

$$\begin{bmatrix} x|a|^2 + w|b|^2 + y|c|^2 + z|d|^2 & x|b|^2 + w|a|^2 + y|d|^2 + z|c|^2 \\ x|c|^2 + w|d|^2 + y|a|^2 + z|b|^2 & x|d|^2 + w|c|^2 + y|b|^2 + z|a|^2 \end{bmatrix}. \tag{8}$$

Thus, the requirement $A^T = B$ is equivalent to constraints expressed by the following system of equations:

$$\begin{aligned} (|b|^2 - |c|^2)(x - z) = 0 \\ (|b|^2 - |c|^2)(y - w) = 0. \end{aligned} \tag{9}$$

We see that when $|b| = |c|$ these constraints are fulfilled whatever are values of payoffs.

On the other hand, if $x \neq z$ or $y \neq w$, then the condition $|b| = |c|$ is a necessary condition to make (9) satisfied, which finishes the proof. □

From this point on we shall assume that an intervention of the arbiter into a game is such that the game remains symmetric. In such case we get the following:

Theorem 3 *A cheating quantum arbiter can, retaining symmetry of the original* 2×2 *game in which no payoffs are identical, change its payoff bi-matrix without changing its category, as well as, in almost all cases, change its category into any other one.*

Proof Of course only the initial state $|\psi_{in}\rangle = a|00\rangle$ with $|a| = 1$ causes no changes in the original bi-matrix (2) of a game.

Let us denote $\alpha_0 = x - w$ and $\beta_0 = z - y$ characteristic numbers of the initial game. Then, assuming that $|b| = |c|$, one can calculate from the matrix (7) of the final game its characteristic numbers

$$\alpha = \alpha_0(|a|^2 - |b|^2) + \beta_0(|d|^2 - |b|^2), \tag{10}$$

$$\beta = \alpha_0(|d|^2 - |b|^2) + \beta_0(|a|^2 - |b|^2). \tag{11}$$

Let us assume that the initial game belongs to category I, i.e., that $\alpha_0\beta_0 < 0$. Then if we put $|a|^2 < |b|^2 = |c|^2 < |d|^2$ in the initial state (6), we get $\alpha\beta < 0$ so the final game also belongs to category I. If the initial game belongs to category II or III, one can easily check that the condition $|b|^2 < \min(|a|^2, |d|^2)$ is sufficient to keep category of the game unchanged.

Since a 'cheating arbiter', while preparing the initial state (6) of a game is constrained only by the normalization condition and, if he wants to retain symmetry of a game, by the condition $|b| = |c|$, it is obvious that in general he can choose coefficients $a, b, c,$ and d in the initial state (6) in such a way that category of a game changes into any other one according to his will. For example, if he prepares the initial state (6) so that $|b|^2 > \max(|a|^2, |d|^2)$, then he changes category of a game from II to III and vice versa. It can be also checked that if $\alpha_0 \neq \pm\beta_0$ and the initial state (6) is such that $|a|^2 \neq |d|^2$ and $|b|^2 = 1/4$, then a game changes its category from II or III to I, and the opposite change is forced by putting in (6) $|a|^2 = |d|^2 \neq |b|^2$.

However, not in every case any conceivable change is possible. One can check, for example, that if $\alpha_0 = \beta_0$ and the original game belongs to category II (resp. III), then it can be changed into a game that belongs to category III (resp. II), but not into a game that belongs to category I. Even worse situation occurs when the original game belongs to category I and $\alpha_0 = -\beta_0$: In this case an arbiter cannot change category of a game.

However, such cases are exceptional and in general a 'cheating arbiter' can change category of a symmetric 2×2 quantum game into any other one according to his will, so the proof is finished. □

6 Concluding Remarks

Our case studies of the influence of a 'cheating arbiter' on symmetric 2×2 games confirm the general belief that the range of possibilities in the 'quantum' case is always much wider than in the 'classical' case. Moreover, extreme sensitivity of 2×2 quantum games on the specific form of the initial state of a pair of qubits used to play a game (called in [3] 'a strategy') suggests changing the way of looking at quantum games. Classical static two-person games are completely defined by their matrices of payoffs and this way of looking at quantum games was up to now adopted also in the quantum domain. Of course it was noticed already in the very first papers on quantum games that players who play a quantum game should base their decisions not only on the analysis of a payoff matrix of a game but also on an initial state of a pair of qubits used as carriers of information in a game, however, it was never considered who and according to what rules prepares this initial state. Since two quantum games that have the same payoff matrices but begin with different initial states of quantum objects used as carriers of information usually force the players to choose different strategies, in our opinion such two quantum games are in fact different games. Therefore, we argue that static quantum games, contrary to their classical prototypes are not fully defined by specifying their payoff matrices alone and defining them requires also defining the specific initial state of a set of quantum objects used to play a quantum game.

Finally, let us note that although players that play a classical static game can exchange information about chosen strategies using for this purpose various physical objects, usually they can do it also verbally or in writing. This way of playing quantum games, except of a possibility of playing 2×2 quantum games according to Marinatto-Weber scheme with the use of macroscopic objects [8] is still, in general, not known.

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